JHEP HYPER VERSION

Received: *February 2, 2001*, Accepted: *March 1, 2001*
Revised: *February 28, 2001*

Gross-Ooguri phase transition at zero and finite temperature: two circular Wilson loop case

Hungsoo Kim,[a] **DaeKil Park**[abc*], **Sayat Tamaryan**[cd], **Harald J.W. Müller-Kirsten**[c]

[a] *Department of Physics, Kyungnam University Masan, 631-701, Korea*
[b] *Department of Physics, Brookhaven National Laboratory Upton, NY 11973-5000, USA*
[c] *Department of Physics, University of Kaiserslautern D-67653 Kaiserslautern, Germany*
[d] *Theory Department, Yerevan Physics Institute Yerevan–36, 375036, Armenia*
Email: hskim@hep.kyungnam.ac.kr, dkpark@quark.phy.bnl.gov, sayat@moon.yerphi.am, mueller1@physik.uni-kl.de

Abstract: In the context of AdS/CFT correspondence the two Wilson loop correlator is examined at both zero and finite temperatures. On the basis of an entirely analytical approach we have found for Nambu-Goto strings the functional relation $dS_c^{(\mathrm{Reg})}/dL = 2\pi k$ between Euclidean action S_c and loop separation L with integration constant k, which corresponds to the analogous formula for point-particles. The physical implications of this relation are explored in particular for the Gross-Ooguri phase transition at finite temperature.

Keywords: Duality in Gauge Field Theories, AdS-CFT Correspondance.

JHEP03(2001)003

*On leave from Kyungnam University, Korea

Recently much attention has been paid to the Wilson loop correlator in large-N gauge theory. This interest results mainly from the fact that AdS/CFT duality [1, 2] makes it more tractable to understand this highly nontrivial quantum field theory effect through a classical description of the string configuration in the AdS background. Using this AdS/CFT duality Maldacena [3] was able to calculate for the first time the expectation value of the rectangular Wilson loop operator at zero temperature and found that the interquark potential exhibits the Coulomb type behavior expected from conformal invariance of the gauge theory, as well as indications of the screening of the charge. Furthermore Witten [4] has shown that the AdS/CFT duality can be used to explore finite temperature behavior of gauge theory by compactifying the AdS euclidean time on a circle of radius $\propto 1/\text{temperature}$.

Maldacena's computational technique has already been extended to the finite temperature case by replacing the AdS metric by a Schwarzschild-AdS metric [5, 6] which implies the same boundary conditions [7]. The main differences of the finite temperature case from the zero temperature one are (1) the presence of a maximal separation distance between quarks, and (2) the appearance of a cusp (or bifurcation point) in the plot of interquark potential-vs-interquark distance. These differences strongly suggest that there is a hidden functional relation between physical quantities, and indeed such a relation has been derived explicitly in ref. [8].

Here we show that there is a similar relation in the two Wilson loop case. It is known [9] that the correlation function of two circular Wilson loops implies a phase transition analogous to that between the catenoid as minimal solution of the area connecting two concentric circles [10] and the associated discontinuous Goldschmit solution. In fact, this Gross-Ooguri(GO) phase transition takes place due to the instability of the classical string configuration. The GO phase transition at zero temperature has been examined in more detail in ref. [11] by solving the equations of motion of the Nambu-Goto string action.

In the following we approach the GO phase transition at zero temperature entirely analytically, which enables us to derive a functional relation $dS_c^{\text{Reg}}/dL = 2\pi k$ between the regularized euclidean classical action S_c^{Reg} and the separation L of the Wilson loops, k being a constant of integration. We will also approach the GO phase transition at finite temperature and examine the physical implications of the functional relation at the finite temperature phase transition. Some related finite temperature aspects have been considered in ref. [12]

We use the classical Nambu-Goto action for a string world sheet

$$S_{NG} = \int d\tau d\sigma \sqrt{det g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu}\,, \tag{1}$$

where $g_{\mu\nu}$ is the euclidean $AdS_5 \times S^5$ metric. We want to study the theory of a type IIB string with coordinates $X^\mu(\tau,\sigma)$ which ends on the near extremal D3-brane, as the dual of an SU(N) gauge theory in the sense of refs. [1, 3], the case of large N, large

JHEP03(2001)003

't Hooft coupling in the gauge theory corresponding to weak coupling perturbative string theory in the semiclassical approximation, and so to supergravity expanded about an $AdS_5 \times S^5$ background. The quantity under the square root is the induced metric on the world sheet of the Nambu-Goto string.

The near extremal euclidean Schwarzschild-AdS_5 metric of a D3-brane (with constant dilaton, and the S^5 coordinates eliminated) [13] is given in Poincaré coordinates by

$$ds_E^2 = \left[U^2 \left(f(U)dt^2 + \sum_{i=1}^{3} dx_i dx_i\right) + \frac{f(U)^{-1}}{U^2} dU^2\right]. \tag{2}$$

Here t is euclidean time of target space, and we choose $R_{ads} = \alpha' = 1$ for simplicity, where R_{ads} and α' are the radius of AdS_5 and Regge slope respectively [1]. U is the holographic coordinate [1]. The function $f(U)$ is given by $f(U) = 1 - U_T^4/U^4$ where U_T^2 is a parameter proportional to the energy density above extremality on the brane [1]; U_T is proportional to the external temperature T defined by $T = U_T/\pi R_{ads}$ [6] which enters through the periodic identification of $t \to t + 1/T$ to make the horizon at $U = U_T$ regular [9]. The world volume coordinates t, x_i of the brane can be regarded as coordinates of the dual $3+1$ dimensional gauge theory.

For simplicity again, we choose cylindrical coordinates $dx_i dx_i = dx^2 + dr^2 + r^2 d\phi^2$, which results in

$$ds_E^2 = \frac{1}{z^2}\left[(1 - U_T^4 z^4)dt^2 + \frac{dz^2}{1 - U_T^4 z^4} + dx^2 + dr^2 + r^2 d\phi^2\right], \tag{3}$$

where $z \equiv 1/U$. Thus the AdS boundary is located at $z = 0$.

For reasons of symmetry we make the following *ansatz* for the minimal surface [11]

$$t = 0\,, \qquad \phi = \sigma\,, \qquad x = x(\tau)\,, \qquad r = r(\tau)\,, \qquad z = z(\tau)\,. \tag{4}$$

Then (with elements $g_{\mu\nu}$ of eq. (3)) the Nambu-Goto action becomes

$$S_{NG} = 2\pi \int d\tau \frac{r}{z^2}\sqrt{\mathcal{L}_\mathcal{T}}\,, \tag{5}$$

where

$$\mathcal{L}_\mathcal{T} = x'^2 + r'^2 + \frac{z'^2}{1 - U_T^4 z^4} \tag{6}$$

and the prime denotes differentiation with respect to τ. The Euler-Lagrange equations derived from action (5) are

$$\begin{aligned} \frac{r}{z^2}\frac{x'}{\sqrt{\mathcal{L}_\mathcal{T}}} &= k\,, \\ \frac{d}{d\tau}\left[\frac{rr'}{z^2\sqrt{\mathcal{L}_\mathcal{T}}}\right] - \frac{\sqrt{\mathcal{L}_\mathcal{T}}}{z^2} &= 0\,, \\ \frac{d}{d\tau}\left[\frac{rz'}{z^2\sqrt{\mathcal{L}_\mathcal{T}}(1 - U_T^4 z^4)}\right] + \frac{2r\sqrt{\mathcal{L}_\mathcal{T}}}{z^3} - \frac{2U_T^4 r z z'^2}{\sqrt{\mathcal{L}_\mathcal{T}}(1 - U_T^4 z^4)^2} &= 0\,, \end{aligned} \tag{7}$$

where k is the integration constant arising from the equation of motion for x.

JHEP03(2001)003

With a gauge choice $\tau = x$ the equations of motion take the form

$$\begin{aligned} r'^2 + \frac{z'^2}{1-U_T^4 z^4} + 1 &= \frac{r^2}{k^2 z^4}\,, \\ r'' - \frac{r}{k^2 z^4} &= 0\,, \\ \frac{d}{d\tau}\left[\frac{z'}{1-U_T^4 z^4}\right] + \frac{2r^2}{k^2 z^5} - \frac{2U_T^4 z^3 z'^2}{(1-U_T^4 z^4)^2} &= 0\,. \end{aligned} \tag{8}$$

We now assume that two circular Wilson loops are located at $x = \pm L/2$ on the AdS boundary and are linked by propagation of the closed string in the bulk of $AdS_5 \times S^5$. Then we have to solve eq. (8) with the boundary conditions

$$\begin{aligned} r(-L/2) &= r(L/2) = R\,, \\ z(-L/2) &= z(L/2) = \epsilon \approx 0\,. \end{aligned} \tag{9}$$

Here R and L are respectively the radius of the circular Wilson loops and the distance between them. We also introduced the positive infinitesimal constant ϵ for the regularization of the minimal surface area later.

We first consider the zero temperature case ($U_T = 0$). This case has been analyzed in ref. [11] partly analytically. Here we approach this case entirely analytically using various kinds of elliptic functions. This completely analytical approach enables one to derive a hidden functional relation explicitly.

For $U_T = 0$ eqs. (8) become

$$\begin{aligned} r'^2 + z'^2 + 1 - \frac{r^2}{k^2 z^4} &= 0\,, \\ r'' - \frac{r}{k^2 z^4} &= 0\,, \\ z'' + \frac{2r^2}{k^2 z^5} &= 0\,, \end{aligned} \tag{10}$$

and after some manipulations can be shown to yield the equation $r^2 + z^2 + x^2 = a^2$, which is solved by

$$r = \sqrt{a^2 - x^2}\cos\theta\,, \qquad z = \sqrt{a^2 - x^2}\sin\theta\,, \tag{11}$$

where $a^2 \equiv R^2 + L^2/4$, and θ obeys

$$\theta' = \pm\frac{a}{a^2 - x^2}\sqrt{\frac{\cos^2\theta}{k^2 a^2 \sin^4\theta} - 1}\,. \tag{12}$$

Here we take the upper sign for $x \in [-L/2, 0]$ and the lower sign for $x \in [0, L/2]$. The integration over θ can be handled by setting $t = \cos^2\theta$. Then

$$\begin{aligned} I(\theta) &\equiv \int_0^\theta d\theta \frac{\sin^2\theta}{\sqrt{\cos^2\theta - k^2 a^2 \sin^4\theta}} \\ &= \frac{1}{2ak}\int_t^1 dt \frac{\sqrt{1-t}}{\sqrt{-t(t-\beta_+)(t-\beta_-)}}\,, \end{aligned} \tag{13}$$

where

$$\beta_\pm = \frac{(2k^2a^2+1) \pm \sqrt{1+4k^2a^2}}{2k^2a^2}. \qquad (14)$$

Using the integral formula [14]

$$\int_t^b dt \sqrt{\frac{b-t}{(a-t)(t-c)(t-d)}} = (a-b)g\left[\Pi(\phi,\alpha^2,\kappa) - F(\phi,\kappa)\right], \qquad (15)$$

where $a > b \geq t \geq c > d$, and

$$\begin{aligned} \kappa &= \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}, \qquad & g &= \frac{2}{\sqrt{(a-c)(b-d)}}, \\ \alpha &= \sqrt{\frac{b-c}{a-c}}, \qquad & \phi &= \sin^{-1}\sqrt{\frac{(a-c)(b-t)}{(b-c)(a-t)}}, \end{aligned} \qquad (16)$$

and Π and F are elliptic integrals of the third and first kinds respectively, one can show that

$$\begin{aligned} I(\theta) = \frac{1}{ka}\frac{\beta_+ - 1}{\sqrt{\beta_+ - \beta_-}} &\left[\Pi\left(\sin^{-1}\sqrt{\frac{(\beta_+ - \beta_-)(1-\cos^2\theta)}{(1-\beta_-)(\beta_+ - \cos^2\theta)}}, \frac{1-\beta_-}{\beta_+ - \beta_-}, \kappa\right) \times \right. \\ &\left. \times F\left(\sin^{-1}\sqrt{\frac{(\beta_+ - \beta_-)(1-\cos^2\theta)}{(1-\beta_-)(\beta_+ - \cos^2\theta)}}, \kappa\right)\right] \end{aligned} \qquad (17)$$

with

$$a = \beta_+, \qquad b = 1, \qquad c = \beta_-, \qquad d = 0,$$

and

$$\kappa = \sqrt{\frac{\beta_+(1-\beta_-)}{\beta_+ - \beta_-}}. \qquad (18)$$

One may note that for $k \to 0$ we have $\beta_- = 0$ and so $\kappa \to 1$.

Using eq. (17) one can integrate eq. (12) completely. The final result for θ assumes the form

$$I(\theta) = \frac{1}{2ka}\ln\frac{(a+L/2)(a \pm x)}{(a-L/2)(a \mp x)}, \qquad (19)$$

where the upper and lower signs correspond again to $x \in [-L/2, 0]$ and $x \in [0, L/2]$ respectively.

From eqs. (12) and (19) one can show easily that $\theta(-L/2) = \theta(L/2) = 0$ and $\theta_0 \equiv \theta(0) = \cos^{-1}\sqrt{\beta_-}$, where the latter is the result of $\theta'(0) = 0$. Inserting $x = 0$ in eq. (19) and realizing that $\sin^{-1}\sqrt{\frac{(\beta_+ - \beta_-)(1-\cos^2\theta)}{(1-\beta_-)(\beta_+ - \cos^2\theta)}} \to \frac{\pi}{2}$ at this point, one can derive

$$\mathcal{F} = \frac{1}{2}\ln\frac{a+L/2}{a-L/2} = \ln\frac{\sqrt{R^2 + (L^2/4)} + (L/2)}{R}, \qquad (20)$$

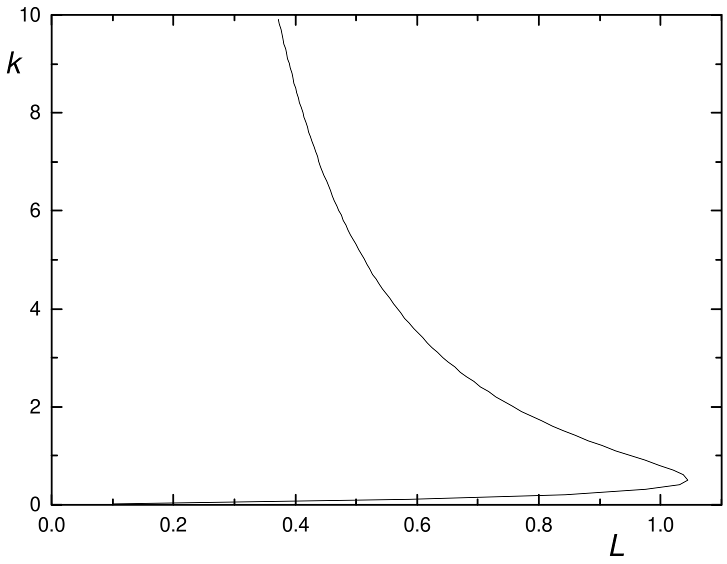

Figure 1: The L–dependence of k at zero temperature showing the two–branch structure for L less than the maximum value L_*.

where

$$\mathcal{F} = \frac{\beta_+ - 1}{\sqrt{\beta_+ - \beta_-}} \left[\Pi\left(\frac{1-\beta_-}{\beta_+ - \beta_-}, \kappa \right) - K(\kappa) \right] \tag{21}$$

and Π and K are complete elliptic integrals.

Hence the k-dependence of L is obtained by solving

$$L = (2\sinh \mathcal{F})R \tag{22}$$

numerically. Correspondingly the equation can be solved for L as a function of k. Figure 1 shows this L-dependence of k when $R = 1$. Figure 1 exhibits the two-branch structure described in ref. [11] and hence shows that there is a maximal distance L_* for the existence of the classical catenoid solution. If $L > L_*$, the classical catenoid solution becomes unstable and hence the physically relevant solution in this case becomes two discontinuous one Wilson loop solutions, which are the so-called Goldschmit discontinuous solutions [9].

The minimal surface is computed by evaluating the action (5) for the classical solution given by eqs. (10) to (12). One obtains

$$\begin{aligned} S_c &= 2\pi \int_{-L/2}^{L/2} dx \frac{r}{z^2} \sqrt{1 + (r')^2 + (z')^2} \\ &= 4\pi \int_{\frac{\epsilon}{R}}^{\theta_0} d\theta \frac{\cot^2\theta}{\sqrt{\cos^2\theta - k^2 a^2 \sin^4\theta}} \\ &= \frac{2\pi}{ak} \int_{\beta_-}^{\cos^2 \epsilon/R} \frac{dt}{(t-1)} \frac{\sqrt{-t}}{\sqrt{(\beta_- - t)(t-1)(t-\beta_+)}} \end{aligned} \tag{23}$$

with angular cutoff ϵ/R. In a way similar to above we use the integral formulas [15]

$$\int_t^b \frac{dt}{t-c}\sqrt{\frac{a-t}{(b-t)(t-c)(t-d)}} = \frac{a-b}{b-c} g \int_0^{u_1(t)} nc^2 u du \,,$$
$$\int nc^2 u du = \frac{1}{\kappa'^2}\left[\kappa'^2 u - E(\phi,\kappa) + dnu\ tnu\right], \qquad (24)$$

where $a < b \leq t \leq c < d$ with $a = 0,\ b = \beta_-,\ c = 1,\ d = \beta_+$, and

$$g = \frac{2}{\sqrt{(a-c)(b-d)}}, \qquad \kappa^2 \equiv 1 - \kappa'^2 = \frac{(b-c)(a-d)}{(a-c)(b-d)},$$
$$\phi = \sin^{-1}\sqrt{\frac{(a-c)(b-t)}{(b-c)(a-t)}}, \qquad u_1(t) = sn^{-1}[\sin\phi] = sn^{-1}\sqrt{\frac{(\beta_- - t)}{t(\beta_- - 1)}}. \qquad (25)$$

In eq. (24) the integration is from $u_1(\beta_-) = 0$ to $u_1(\cos^2\epsilon/R) \simeq sn^{-1}(1 - \frac{\beta_-}{2(\beta_- -1)}\frac{\epsilon^2}{R^2})$. Then one can evaluate eq. (23) which is

$$S_c = \frac{4\pi R}{\epsilon} + S_c^{(\mathrm{Reg})},$$
$$S_c^{(\mathrm{Reg})} = 4\pi(1+4k^2a^2)^{1/4}\left[(1-\kappa^2)K(\kappa) - E(\kappa)\right], \qquad (26)$$

where κ is the same as that of eq. (18) and $E(0,\kappa) = 0, E(\pi/2,\kappa) \equiv E(\kappa), E(1) = 1$. The divergent part of S_c results from $dnu\ tnu$ in eq. (24) and is obtained by expanding tnu around $u = K$, i.e. with $tnu = snu/cnu \simeq -1/(u-K)\kappa'$. Here $u-K$ is obtained in terms of ϵ/R by expanding snu around $u = K$ which gives $1-(u-K)^2\kappa^2/2$ and by then comparing with $snu_1(\cos^2\epsilon/R)$ above. Thus $tnu|_{u\to K} = \sqrt{\frac{1-\beta_-}{\beta_-}}\frac{R}{\epsilon}$. Multiplying by the factors in front of the integral one obtains $4\pi R/\epsilon$.

The area of the discontinuous solution has been calculated in ref. [16] using the special conformal transformation. Of course, this can also be evaluated directly by considering the case of one circular Wilson loop. Then the minimal surface of the discontinuous solution is

$$S_{dc} = \frac{4\pi R}{\epsilon} + S_{dc}^{(\mathrm{Reg})},$$
$$S_{dc}^{(\mathrm{Reg})} = -4\pi\,. \qquad (27)$$

One should note that for $k = 0$ the expression $S_c^{(\mathrm{Reg})}$ is identical with $S_{dc}^{(\mathrm{Reg})}$. This is expected from the first of eqs. (7) which indicates that there is no propagation of the string along x when $k = 0$.

Figure 2 shows the L-dependence of $S_c^{(\mathrm{Reg})}$ and $S_{dc}^{(\mathrm{Reg})}$ and demonstrates the phase transition at a critical value $L = L_*$. One should note that $S_c^{(\mathrm{Reg})}$ merges smoothly with $S_{dc}^{(\mathrm{Reg})}$ at $L = 0$, which indicates again that $S_c^{(\mathrm{Reg})} = S_{dc}^{(\mathrm{Reg})}$ at $k = 0$. The appearance of the cusp in $S_c^{(\mathrm{Reg})}$ at $L = L_*$ strongly suggests that there is a hidden functional relation [8] in the case of the two Wilson loop correlator.

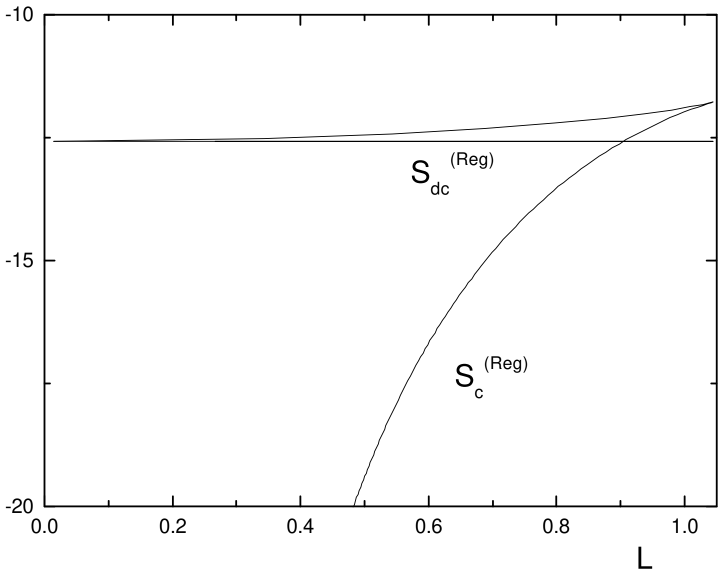

Figure 2: L-dependence of $S_c^{(reg)}$ and $S_{dc}^{(reg)}$ at zero temperature demonstrating the appearance of the cusp at the critical value of L which is the largest value of L in figure .

In order to derive this relation explicitly we differentiate $S_c^{(\mathrm{Reg})}$ and L with respect to the elliptic modulus κ. This is straightforwardly achieved using the various derivative formulas of elliptic functions [14]. The final relations are simply

$$\begin{aligned} \frac{dS_c^{(\mathrm{Reg})}}{d\kappa} &= -\frac{4\pi\kappa}{(2\kappa^2-1)^{\frac{3}{2}}}\left[K(\kappa)-2E(\kappa)\right], \\ \frac{dL}{d\kappa} &= -\frac{2a}{\kappa'\sqrt{2\kappa^2-1}}\left[K(\kappa)-2E(\kappa)\right]. \end{aligned} \tag{28}$$

One should note that the coefficients of the complete elliptic integrals in the brackets coincide with each other, which thus yields after some amazing cancellations

$$\frac{dS_c^{(\mathrm{Reg})}}{dL} = 2\pi k\,. \tag{29}$$

As observed in ref. [8], this relation has a close analogy with the point-particle formula $dS_E/dP = \mathcal{E}$, where S_E, P and $\mathcal{E}$ are euclidean action, period and energy of the classical point-particle. It is worthwhile noting that eq. (29) and a condition $S_c^{(\mathrm{Reg})} = S_{dc}^{(\mathrm{Reg})}$ at $L=0$ determine completely the L-dependence of $S_c^{(\mathrm{Reg})}$ from the k-dependence of L since eq. (29) says that $S_c^{(\mathrm{Reg})} = 2\pi\int kdL$ up to the constant. We expect a formula of this type to appear in many models and thus to have some universal validity.

We now turn to the GO phase transition at finite temperature. In this case it seems impossible to attack eqs. (8) analytically. However, the analysis of the zero temperature case shows how one can solve eqs. (8) numerically. In fact, one can

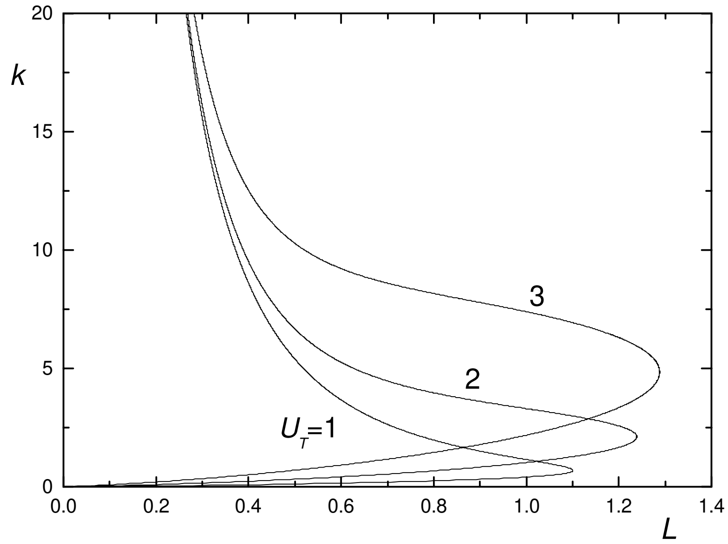

Figure 3: The L-dependence of k at finite temperatures $\propto U_T = 1, 2, 3$.

solve the second and third of eqs. (8) simultaneously using the first equation as a relation of boundary conditions at $x = 0$. Solving these coupled differential equations completely determines L and R. If R is fixed by R_0, the only step that remains is to select the solutions which yield $R = R_0$.

Figure 3 shows the L-dependence of k at finite temperature when $R = 1$. Figure 3 indicates that the peak point moves to the upward, and the maximum distance of the Wilson loop L_* becomes larger when the temperature increases.

Next we consider the minimal surface area in the finite temperature case. For this quantity the numerical approach is not a useful tool in view of the divergent term which arises in the course of the calculation. It is, in fact, a formidable task to regularize the minimal surface in the numerical technique. However, assuming that eq. (29) also holds in the finite temperature case, one can conjecture the L-dependence of $S_c^{(\mathrm{Reg})}$ from figure 3 since eq. (29) tells us that $S_c^{(\mathrm{Reg})} = 2\pi \int k dL$ up to a constant. To fix this constant we need $S_{dc}^{(\mathrm{Reg})}$ of the finite temperature case.

However, even with the numerical technique the computation of $S_{dc}^{(\mathrm{Reg})}$ is not an easy problem — again in view of the divergent term. In order to evaluate $S_{dc}^{(\mathrm{Reg})}$ at finite temperatures we observe that the minimal surface becomes

$$S_c = 4\pi k \left[rr' \,|_{x=L/2} - \int_0^{L/2} dx r'^2 \right]. \tag{30}$$

This is obtained from eq. (5) and eq. (8) and by performing an appropriate partial integration. If one examines the behavior of $r(x)$ and $z(x)$ as $x \to L/2$, it is easy to show that the second term in eq. (30) is finite. Hence the divergence of S_c is contained in the first term of eq. (30). In the zero temperature case, for example,

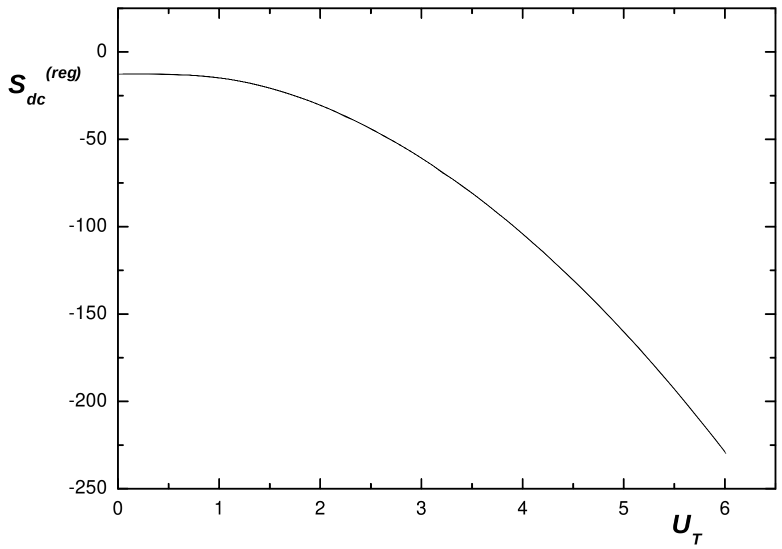

Figure 4: The temperature–dependence of $S_{dc}^{(reg)}$.

one can derive the asymptotic behavior of $r(x)$ and $z(x)$ for $x \approx L/2$:

$$
\begin{aligned}
r &\approx R - \frac{1}{2R}\left(\frac{3R}{kL}\right)^{2/3} y^{4/3} + \frac{y^2}{2R} + \cdots\,, \\
z &\approx \left(\frac{3R}{kL}\right)^{1/3} y^{2/3} - \frac{3}{5RkL} y^2 + \frac{1}{2R^2}\left(\frac{3R}{kL}\right)^{1/3} y^{8/3} + \cdots\,,
\end{aligned}
\tag{31}
$$

where $y^2 \equiv L^2/4 - x^2$. It is important to note that the coefficient of y^2 in r is not determined by direct expansion but from the relation $r^2 + z^2 + x^2 = a^2$, which does not have a counterpart in the finite temperature case. Hence in the zero temperature case $rr' \mid_{x=L/2}$ becomes

$$
rr' \mid_{x=L/2} = \frac{R}{k\epsilon} - \frac{L}{2} \tag{32}
$$

which yields same divergence term $4\pi R/\epsilon$ and $S_c^{(\mathrm{Reg})} = -2\pi kL - 4\pi k \int_0^{L/2} dx r'^2$. If one plots this numerically, one can reproduce $S_c^{(\mathrm{Reg})}$ in figure 2. The important point one should note is that the finite term in $4\pi k r r' \mid_{x=L/2}$, i.e. $-2\pi kL$, becomes zero in the limit $k \to 0$. Thus the limit $k \to 0$ of $S_c^{(\mathrm{Reg})}$, which is nothing but $S_{dc}^{(\mathrm{Reg})}$, originates from the second term of eq. (30). We believe this property is maintained in the finite temperature case. Then one can plot the temperature-dependence of $S_{dc}^{(\mathrm{Reg})}$ which is shown in figure 4. Figure 4 completely determines the shift constant in $S_c^{(\mathrm{Reg})} = 2\pi \int k dL$. Figure 5 shows $S_c^{(\mathrm{Reg})}$ at various temperatures. As explained, the appearance of the cusp in $S_c^{(\mathrm{Reg})}$ indicates the non-monotonic behaviour of the k-dependence of L or L-dependence of k. A similar cusp was observed in ref. [17] in a different calculation. In figure 6 we show the temperature-dependence of the critical point L_* at which the phase transition occurs. In the same plot we also show

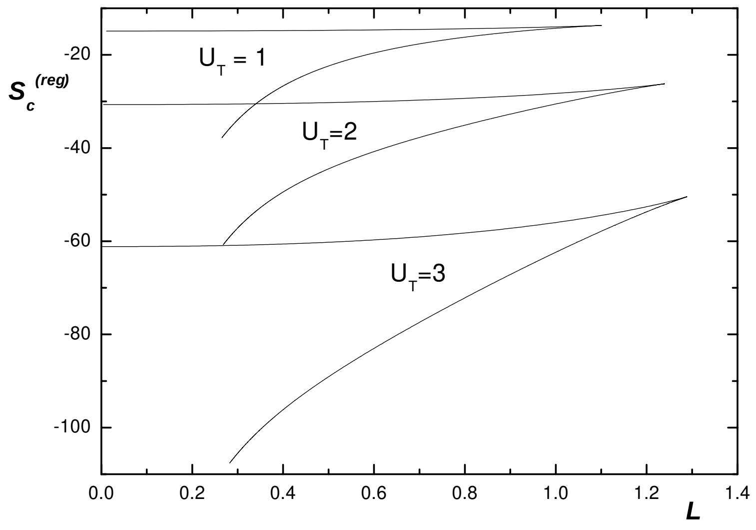

Figure 5: L–dependence of $S_c^{(reg)}$ at finite temperatures.

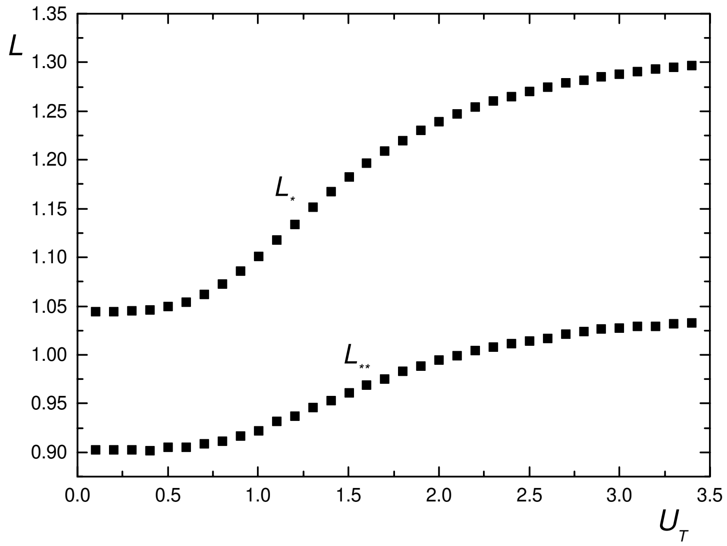

Figure 6: The plot marked L_* shows the temperature dependence of the bifurcation point at which the phase transition occurs. The plot marked L_{**} shows the temperature dependence of the point L at which the action has the same value as that of the discontinuous solution.

the temperature-dependence of the point at which the action has the same value as that of the discontinuous solution.

In summary, we analyzed above the Gross-Ooguri phase transition at zero and finite temperatures. We obtained the functional relation $dS_c^{(\mathrm{Reg})}/dL = 2\pi k$ which is the Nambu-Goto string analogue of the formula relating euclidean action to period and energy of a classical point-particle [18]. It is interesting to investigate this point-particle analogy in more detail by calculating the spectrum of the fluctuation operator. This work is in progress.

Acknowledgments

D.K.P. acknowledges support by the Korea Research Foundation Grant (KRF–2000–D00073).

References

[1] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rep.* **323** (2000) 183 [hep-th/9905111].

[3] J. Maldacena, *Wilson loops in large-N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002].

[4] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].

[5] S.-J. Rey, S. Theisen and J.-T. Yee, *Wilson-Polyakov loop at finite temperature in large-N gauge theory and anti-de Sitter supergravity*, *Nucl. Phys.* **B 527** (1998) 171 [hep-th/9803135];
S.-J. Rey and J. Yee, *Macroscopic strings as heavy quarks in large-N gauge theory and anti-de Sitter supergravity*, hep-th/9803001.

[6] A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, *Wilson loops in the large-N limit at finite temperature*, *Phys. Lett.* **B 434** (1998) 36 [hep-th/9803137].

[7] S.W. Hawking and D.N. Page, *Thermodynamics of black holes in anti-de sitter space*, *Comm. Math. Phys.* **87** (1983) 577.

[8] D.K. Park, H.J.W. Muller-Kirsten and A.V. Shurgaia, *Functional relation of interquark potential with interquark distance*, *Phys. Lett.* **B 501** (2001) 54 [hep-th/0008224].

[9] D.J. Gross and H. Ooguri, *Aspects of large-N gauge theory dynamics as seen by string theory*, *Phys. Rev.* **D 58** (1998) 106002 [hep-th/9805129].

[10] H. Goldstein, *Classical Mechanics*, Addison-Wesley ed. , Reading 1950.

[11] K. Zarembo, *Wilson loop correlator in the AdS/CFT correspondence*, *Phys. Lett.* **B 459** (1999) 527 [hep-th/9904149];
P. Olesen and K. Zarembo, *Phase transition in wilson loop correlator from AdS/CFT correspondence*, hep-th/0009210.

[12] J. Greensite and P. Olesen, *Broadening of the QCD_3 flux tube from the AdS/CFT correspondence*, *J. High Energy Phys.* **11** (2000) 030 [hep-th/0008080].

[13] G.T. Horowitz and A. Strominger, *Black strings and p-branes*, *Nucl. Phys.* **B 360** (1991) 197.

[14] P.F. Byrd and M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, formula 251.05., Springer, Berlin 1971.

[15] Ref. [14], formula 251.15.

[16] D. Berenstein, R. Corrado, W. Fischler and J. Maldacena, *The operator product expansion for Wilson loops and surfaces in the large-N limit*, *Phys. Rev.* **D 59** (1999) 105023 [hep-th/9809188].

[17] H. Dorn and H.J. Otto, *$Q\bar{Q}$ potential from AdS-CFT relation at $T \geq 0$: dependence on orientation in internal space and higher curvature corrections*, *J. High Energy Phys.* **09** (1998) 021 [hep-th/9807093].

[18] J.Q. Liang, H.J.W. Müller-Kirsten, D.K. Park and F. Zimmerschied, *Periodic instantons and quantum classical transitions in spin systems*, *Phys. Rev. Lett.* **81** (1998) 216;
S.Y. Lee, H.J.W. Müller-Kirsten, D.K. Park and F. Zimmerschied, *Quantum tunneling and phase transitions in spin systems with an applied magnetic field*, *Phys. Rev.* **B 58** (1998) 5554.